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Laine, Anu

2018-06

Laine , A , Näveri , L , Ahtee , M , Pehkonen , E & Hannula , M S 2018 , ' Connections of
þÿPrimary Teachers Actions and Pupils Solutions to an Open Problem '
Journal of Science and Mathematics Education , vol. 16 , no. 5 , pp. 967-983 . https://doi.org/10.1007/s10763-017-98

http://hdl.handle.net/10138/235195
https://doi.org/10.1007/s10763-017-9809-3

Connections of Primary Teachers' Actions and Pupils' Solutions for an Open Problem

Anu Laine, Liisa Näveri, Erkki Pehkonen, Maija Ahtee & Markku S. Hannula

Abstract: The aim of this article is to find the connections between teachers' actions (N = 7) and their third-graders' performances (N = 86) when solving an open problem. A lot of information about the teaching sessions was gathered by video recordings and collecting pupils' solutions. Teachers' actions were analysed carefully through the phases of a problem-solving lesson (launch phase, explore phase and discuss and summarize phase). Pupils' solutions levels were categorised and their performances were compared in different teaching groups. Based on our data, teachers' way of teaching varied greatly in every phase. In the launch phase, the way in which the teachers presented the problem and especially how they illustrated the central concepts affected pupils' performances. The teacher's way of guiding and the choice of the equipment was central in the explore phase. However, even though some teachers revealed the central ideas of the solution during the launch and explore phases, this had no effect on the pupils' solutions.

Keywords: open problem, primary mathematics, problem solving, teacher's actions

Introduction

Already at the primary level in Finland, the aim of learning mathematics is to understand mathematical structures, not merely mechanical calculation. The curriculum for the Finnish comprehensive school (National Board of Education, 2004) offers problem solving at all levels as one of the formal objectives for all school subjects. This paper concentrates on the use of problem solving – especially open problems – in mathematics teaching, and especially on how teachers' actions are reflected in pupils' performance in solving an open problem.

Here we will deal with two theoretical constructs that are central for the following empirical study: problem solving and the teacher's role during a problem-solving lesson. We will observe the teacher's role during different phases of the problem-solving lesson.

Problem Solving

Problem solving is understood as a method to develop mathematical thinking (e.g. Schoenfeld, 1985). Problems can be classified as open or closed. In open problems, there might be several alternatives for the starting and/or end situations as well as for the solution method. The most typical open problem is an open-ended problem, where the starting point is given, but there are many possible answers and no unique right answer is predetermined (cf. Pehkonen, 1995; Nohda, 2000). An example of an open-ended problem is as follows: "Divide a rectangle into three triangles. Can you find other solutions?"

Schroeder and Lester (1989) have identified three different approaches to problem solving in mathematics teaching: teaching *about* problem solving, teaching *for* problem solving and teaching *via* problem solving. Teaching about problem solving defines problem solving as a specific strand in the curriculum, and its implementation means solving separate problems regularly. Teaching for problem solving considers problem solving as a tool for e.g. applications. Teaching via prolem solving regards problem solving as a general teaching method rather than a skill to be learned. In this article, we explore the feasibility of the first and last approach and examine how different teacher actions contribute to developing mathematical thinking in the connection of a specific mathematical idea and to learning this idea. We use an open problem that introduces *point symmetry*.

Teacher's Role During the Problem-solving Lesson

Teacher's role is important in all learning. Teacher sets the frames for learning, plans learning environments and thus makes quality learning possible. Why do some teachers seem to be better in teaching problem solving than others? There are few studies looking at the student-teacher interactions when working on open problems. For example, Chiu (2007) reports how three teachers guided fifth graders in fractions using creative and non-creative problems. The best of these three approaches for teaching creative problem solving seemed to be teaching that emphasises a confidential learning environment, self-regulation, imagination and basic understanding.

We will next summarise some key findings in teaching problem solving.One point of view is related to the quality of a teacher's knowledge (Shulman, 1986). Unlike in usual mathematics problems, there is no indication for an open problem to be "finished". Therefore, even a well-prepared teacher may be surprised when students discover powerful mathematical ideas that the teacher has not anticipated. Therefore, open problems require that the teacher has the pedagogical competence to listen to their students' solutions , the content knowledge to recognise students' unexpected mathematical ideas even when they are not fully formulated and the pedagogical content knowledge to guide students when they seek, formulate and communicate their ideas.

For example, Schoenfeld (1992, an adaptation of Lester's work), Sawada (1997), and Stein, Engle, Smith and Hughes (2008) have identified three phases as being typical in problem solving or inquiry lessons. Stein and her colleagues named these phases as 1) a launch phase in which the teacher introduces the problems without giving solution methods or examples, 2) an explore phase in which the students work on problems in small groups under the teacher's guidance and 3) a discuss and summarize phase in which the students present and discuss their solution methods and the teacher summarises the lesson. These three phases can be observed in problem-solving lessons.

The teacher's role in the launch phase. During the launch phase, the teacher presents a task to the class. She must ensure that students understand what they are required to do and the nature of the things they are expected to produce. The teacher should motivate the students so that they are engaged to work on the task. The teacher also has to organise the work, set up the resources and plan the timing for the session. All this needs to be decided when the teacher is planning the lesson (Stein et al., 2008; Sawada, 1997).

Schoenfeld (1992) presents more detailed ideas for teacher actions that give cognitive support for the students. It is important to read the problem and discuss words or phrases students may not understand. It is also central to focus students' attention on important data so that the students can understand the problem and know how to start solving the problem.

A large body of research indicates the importance of affective elements in mathematical problem solving (e.g. Schoenfeld, 1985; Goldin, Epstein, Schorr & Warner, 2011). Although still somewhat inconclusive, it seems that positive emotions would facilitate the creative aspects of problem solving, while modestly negative emotions would facilitate reliable memory retrieval and performance of routines (Pekrun & Stephens, 2010). Hence, one of the ways teachers may facilitate creative problem solving is through creating a happy mood (a proper emotional climate) in their classes during the launch phase.

The teacher's role in the explore phase. In the explore phase students work on the problem, often discussing it in pairs or small groups. As students work, the teacher supports students' autonomous work by encouraging them to solve the problem in whatever way makes sense to them. Here it is important to provide positive yet realistic feedback to support students' self-confidence and engagement with the task (Linnenbrink & Pintrich, 2003). Students may also need help to cope with moments of frustration that are inevitable in problem solving (cf. Hannula, 2015). As students begin to make progress, the teacher should select examples from students' productions to be discussed in the class (Stein et al., 2008). The teacher-student interaction during problem solving has been analysed using a variety of related approaches. Schoenfeld (1992) suggests that teachers should facilitate students' cognition through observing their progress and asking questions (diagnose strengths and weaknesses) and providing hints and problem extension as needed, as well as requiring students who obtain a solution to answer the problem. In addition to such a holistic approach, other researchers have focused on teachers' questions, how they listen to student responses and their feedback to the students.

Teachers need to be careful in the usage of questioning to promote students' reasoning (e.g. Sahin & Kulm, 2008). Anghileri (2006) discusses Wood's critique of the funnelling pattern and focusing as an alternative interaction mode. While funnelling gradually delimits the students' answers until the teacher receives the predetermined response, the focusing-interaction pattern requires the teacher to listen to the students' responses and guide them based on what the students are thinking rather than how the teacher would solve the problem. Pehkonen & Ahtee (2006) have suggested empathic listening as the most advanced level of teacher attending to student ideas. In empathic listening, a teacher strives to understand a student's thoughts and to hear, follow and understand the student's ideas from the student's viewpoint. In such interactions the primary intention is to help students voice their own ideas rather than lead them to the ideas of the teacher.

The last perspective on the interaction focuses on the nature of the teacher's guidance. Based on Hähkiöniemi's and Leppäaho's (2012) three levels of teacher's guidance (surface-level guidance, inactivating guidance and activating guidance) we identify two dimensions for the quality of teacher's guidance: 1) focus on the relevant ideas and 2) activation of student thinking. Firstly, the teacher may either focus to understand the potentially fruitful aspects of the student's idea (deep guidance) or fail to understand the key idea presented by the student and focus on his own idea instead (surface-level guidance). Secondly, the teacher may ask questions that support students' own inventing (activating guidance) or the teacher may reveal the key idea to the students, taking away the possibility for the students's own innovation (inactivating guidance).

The teacher's role in the discussion and summary phase. In Polya's (1945) classical problem-solving model, the *looking back* phase is an important opportunity to learn something new. In the discussion and summary phase, the lesson concludes with a whole-class discussion of students' solutions for the problem (Stein et al., 2008). During this phase, the whole class views and discusses a variety of approaches to the problem (Sawada, 1997). Stein et al. (2008) consider it important that the teacher's comments and responses to the students' questions do not lower the level of cognitive demand of the task. The teacher has to orchestrate the discussion, not just manage the students' interventions and interactions but also promote the mathematical quality of the students' explanations and discussion. The teacher also needs to create and maintain a positive climate of genuine interest in the discussion, trying to guarantee the participation of all the students.

It is very important that the discussion aims beyond the comparison and confrontation of students' solutions. The teacher has a crucial role in helping students to summarise the main mathematical ideas that emerge from the discussion in order to advance the mathematical learning of the whole class. The teacher has the opportunity to focus the students' attention on new concepts and reinforce their mathematical reasoning. The teacher may show and name different strategies, relate ideas to previously solved problems, point to extensions that demonstrate the general applicability of problem-solving strategies and show how specific task features (e.g. chosen visualisations) may influence the problem-solving approach (Schoenfeld, 1992; see also Leinhardt & Schwartz, 1997).

Research Problems

In this paper our aim is to describe seven teachers' actions during problem-solving lessons when they teach the same problem for their pupils. Especially we want to clarify the connections that exist between a teacher’s actions and pupils’ solutions. We focus on the teacher’s actions in the three phases of the teaching model: Launch phase, Explore phase and Discuss and summarize phase. Pupils’ performances will be classified according to the teachers’ behaviour in these phases, and furthermore, their performances will be compared in different teaching groups. Thus, we put forward the following three research problems:

(1) What kind of differences can be found in teachers’ actions during the problem-solving lesson?

(2) What kind of levels can be found in the pupils’ solutions in an open problem?

(3) What factors in the teachers’ actions seem to have an effect for pupils to reach higher-level solutions?

The Empirical Study

This study is part of the three-year Finland–Chile research project, financed by the Academy of Finland (project number #135556, see more e.g. in Pehkonen et al., 2013). Our study included 10 teachers and their third graders from the surroundings of Helsinki. Once a month on average, during the mathematics lessons one open problem was dealt with and recorded on video. Also once a month, the teachers met with the researchers, in order to discuss the implementation of and experiences with the last experimental problem and to introduce the next problem. However, in this study we had to exclude three teachers because they had changed the task so that they asked pupils to divide the square into equal pieces (not in two equal pieces).

The Open Problem Used

Here we consider the following open-ended problem: “Divide a square with a line into two exactly equal pieces.”

The curricular objective for the task is the general one: development of problem-solving skills (cf. National Board of Education, 2004). The concept of symmetry is introduced in two forms: line symmetry and point symmetry. The idea of line symmetry should be familiar to the third graders, whereas point symmetry belongs to the fifth-grade syllabus. Therefore, this task can be used as a preliminary stage in teaching symmetry via problem solving.

When analysing the task more closely, we can find two different aspects:

Use of symmetry. When using the idea of line symmetry you can easily find two basic solutions: a square is divided into two triangles with a diagonal, and into two rectangles with a straight line parallel to the sides. In order to invent more sophisticated solutions you have to give up the idea of line symmetry and understand that the lines have to go through the middle-point.

Nature of the line. The nature of the line can be either straight, curvilinear or a polygonal chain.

Data Gathering

In this study, there were seven teachers (Ann, Beatrice, Cecilia, Danielle, Eve, Fatima and Gabrielle) and a total of 86 pupils from typical primary schools in the Helsinki area. Their classes are average classes and the pupils are not selected for these schools. In Gabrielle's group the pupils were selected from other parallel classes because these pupils had some problems in mathematics.

About two weeks before the problem-solving lessons, the teachers and researchers met in a session were a new problem was given and discussed, but no model solution to the problem was presented. The teachers planned and implemented the lesson in their classrooms according to their own ideas. The observations of the problem-solving lesson and the videos of the lesson were made by two researchers in the classroom; one of the researchers recorded the teacher's actions on video, and the other one focused on the pupils. After the lesson, the researchers collected the teachers' lesson plans and their pupils' solutions. The lesson plans were about half a page long, with the main ideas presented in writing.

All pupils in this project were tested for their calculating, application and problem-solving skills in the beginning of this research (see also Laine, Näveri, Ahtee & Pehkonen, 2014). The differences between the classes were not very large (see Table 3, pre-test).

Analyses of the Data

Pupils' solutions were evaluated by the first two authors in cooperation in the following way: First, all the pupils' solutions were gathered together and the ranking levels of the solutions were discussed and decided. Secondly, the two researchers first looked through the pupils' solutions in one class and classified them into different levels of sophistication. Then the solutions from the other classes were classified separately, and compared. The consensus between the two classifiers was very good (95%).

Furthermore, the two researchers classified in cooperation the teachers' actions (introduction of the task, guidance, summary of the task). The analysis was done from the viewpoint of the whole class. After charting all the found alternatives, the researchers decided the classification. This interpretation made by looking the videos was supported by examining the lesson plans to identify the teachers' objectives. Examples of the coding will be given in the Result section.

Results

We first describe the problem-solving lessons from the perspective of three phases: launch, explore and discuss and summarize phases. In each phase, we try to find out what kind of teachers' actions support pupils' mathematical thinking and would give them a good starting point for understanding a certain topic (point symmetry). After this, we discuss the pupils' performance in solving the open problem and connect the pupils' solution levels with different teachers. In the next stage, we consider the teachers' actions from different viewpoints in order to understand this connection.

Launch Phase

First of all, the pupils have to understand what the given problem means (cf. Polya, 1945). Therefore, it is important to analyse how a teacher poses the task and on which features she focuses the pupils' attention. In the square task (divide a square with a line into two exactly equal pieces), the pupils need to know what the demand "exactly equal pieces" means and how they can check this.

Every teacher began the task introduction by explaining the concept of a square. However, three ways were distinguished in the way they introduced dividing it: No model, Model and Incorrect model.

Only one teacher, Eve, did not present any model, No model, to her pupils. She just showed the wording of the task using the visualiser (projects the image of an object on a screen) and said (confusingly) to her pupils: "Divide a square with a line into two pieces, but the lines may be shapes," probably trying to tell that the line does not have to be straight. She distributed a number of blank squares to the pupils to work with. The pupils had to understand the key features in the task just by themselves.

The second case, Model, represents those cases where the teachers (Ann, Beatrice, Cecilia and Danielle) showed what the division "into two exactly similar pieces" means with some concrete model – a square, circle, triangle, etc.–

Ann and Beatrice used a square as the model. Ann showed a square drawn on a blank sheet of paper. One of the pupils drew a diagonal onto the square using the visualiser. Beatrice for her part cut a square of paper with scissors into two rectangles and stated: "This is possibly the easiest way to divide a square into two parts."

Cecilia showed an isosceles triangle and asked the pupils how it could be divided into two identical parts. After the pupils' responses, she cut the triangle perpendicularly to the base. Danielle began her lesson by drawing a circle on the blackboard. She asked her pupils to show how it could be divided into two similar parts. One pupil drew a line diagonally and another one horizontally through the center of the circle. Then some pupils were shouting "I know a third solution!" "I know a fourth solution!" and so on. Danielle did not allow these pupils to come and draw their solutions. Obviously, many pupils realised that there can be a lot of solutions.

All the teachers explained to their pupils that the correctness of the division can be checked by cutting a square and putting the pieces one on top of the other but only Beatrice and Cecilia cut their model into two identical pieces and showed how it can be done.

Incorrect model represents a situation where the teacher used a misleading model. Two teachers (Fatima and Gabrielle) were classified into this category. Fatima began her lesson by discussing the two basic solutions and by showing them by folding blank sheets. Next she folded a squared paper similar to that given to the pupils many times in different directions, opened it, and said that the pupils can use the foldings as lines when they draw their lines. She thus actually limited her pupils' thinking with her paper-folding model.

Gabrielle brought to her lesson a paper napkin, which she folded first parallel to the sides and then spread it. After that she asked the pupils: "What do you notice from the two parts?" A pupil answered: "Both parts are similar," and Gabrielle continued: "Yes, they are symmetrical." During the lesson Gabrielle repeated many times that "both parts are symmetrical." She even drew a line-symmetric figure on the blackboard, thus restricting her pupils' ideas to line symmetry.

Fatima’s and Gabrielle’s misleading models may have at least two consequences: they may restrict pupils’ mathematical thinking and they do not support the invention of the new idea (point symmetry).

It is important to analyse the launch phase also from the perspective of how the lesson is organised: What tools are available, what kind of social norms are established for interaction and what kind of emotional climate is created? All the teachers gave their pupils scissors and squares cut out of cardboard. In addition, some teachers (Cecilia, Danielle, Eve, Fatima and Gabrielle) gave blank sheets of paper whereas Ann and Beatrice gave grid paper, which helps in properly drawing a point-symmetric case. Later in her lesson Beatrice gave also blank sheets of paper to her pupils. In Ann’s and Eve’s classes pupils worked in pairs and individually in Beatrice’s, Cecilia’s, Danielle’s, Fatima’s and Gabrielle’s classes. However, the individually working pupils were allowed to discuss with others. In all classes the atmosphere was relaxed and safe and all the pupils were excited about the task.

Explore Phase

In the explore phase, our focus of analysis was on how the teachers were guiding their pupils. In our analytical framework we had identified two dimensions: activating vs. inactivating and surface vs. deep guidance. In our analysis we found no inactivating guidance in the teachers’ interactions with individual pupils. Hence, we could identify three modes of support: Activating support (= activating deep guidance), Commenting support (=activating surface guidance) and No support. In activating support the teacher guides with questions based on pupils’ solution. Commenting support means that the teacher is supporting and encouraging the pupils to solve the problem but she is not able to advance the pupils’ solution. No support category contains the teachers who did not guide their pupils at all. Although the teachers did not resort to inactivating guidance in their one-to-one interaction with students, they did reveal key ideas as part of the whole class discussions, which we will discuss later.

Two teachers (Ann and Beatrice) were classified to the category Activating support because they asked many questions that helped the pupils move forward. Ann's pupils worked in pairs and she walked around the classroom. The pupils were restless, and Ann mostly focused on the most restless ones. For Ann the meaning of the middle-point was very clear, and she strived to reach this objective through questioning. She focused on certain pupils' performances and asked essential questions. For example, when showing Peter's solution (see Figure 1) with the visualiser, Ann stated: "Peter is now at the phase that he has cut out the pieces. Are these pieces identical?" Ann followed with her finger the shapes of the pieces from both sides and asked: "How could you make both of these pieces identical?" Peter answered: "Now I know. In the middle you must have as much paper on both sides."

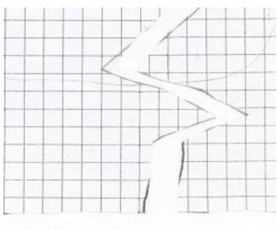

Figure 1. Peter's solution how to divide a square with a line into two equal pieces.

After this event, Carl asked Peter to explain his thinking. Peter went to Carl and, therefore, he did not have time to finish his own solutions. Carl and some other pupils were able to produce point-symmetrical solutions. During the lesson Ann paid no attention to the form of the line in the whole class discussion.

Beatrice's pupils worked individually and she guided them in the direction she wanted by focusing on each pupil's solution: "Think how you could cut so that the pieces would be exactly equal. Take advantage of the grids. You have the right idea." She understood the meaning of the middle-point and therefore she was able to help her pupils with good questions. She analysed one pupil's solution as follows: "You measured the middle-point and put the line a little aslant. Good solution, Matt."

Beatrice's aim was to get her pupils to see that the dividing line is not necessarily straight. She strove for this aim first by using grid paper. To some pupils who used polygonal chains in their solutions, she gave blank paper. Ten minutes from the beginning, she showed a polygonal chain solution and asked all the pupils: "Tell me, what the line looks like?" One pupil answered: "It looks straight." Beatrice continued: "Does the line always have to be straight? No, it does not." She drew a curved line and a polygonal chain on the blackboard. Then she continued: "In the task it was said with a line. Do at least two solutions where the line is not straight." However, about ten minutes before the end of the lesson, using the visualiser she presented her own curved line models without a marked middle-point. In these two instances she used inactivating guidance. Only half of the pupils seemed to be able to develop their own curved line solutions based on Beatrice's models. Beatrice did not reveal the importance of the middle-point in her teaching although she guided pupils into that direction by questioning.

We can conclude that both Ann and Beatrice were able to give good cognitive support for individual pupils i.e. they used activating support. Ann used both cognitive guidance and emotional support in her teaching, such as praising pupils' solutions and encouraging them to continue. Beatrice concentrated mostly on the cognitive parts of the learning. In one to one guidance she used activating questions and comments. However, there were two instances where she revealed an important idea to the whole class.

Commenting support. Four teachers' (Cecilia, Danielle, Eve and Gabrielle) guidance neither concentrated on central features of the problem nor promoted the pupils' thinking; their cognitive support for the pupils was quite modest. Their questions did not help the pupils to proceed better. They just commented: “Carry on thinking.” Instead they concentrated on emotional support, like “Well invented” and “You have had a lot of good thinking.”

Cecilia’s pupils worked individually. She guided her pupils strongly to the direction that she had in her mind. This can be seen, for example, from the fact that as soon as some pupils discovered something essential, Cecilia brought the matter to the attention of the others. However, Cecilia’s questions and guidance did not activate the pupils to think. She just gave an idea like in the following: Early in the lesson she showed Keith’s solution where he had divided the square with a straight line that was neither diagonal nor parallel to the side. She asked: “What is the line like?” One pupil answered: “Like straight.” Cecilia continued: “Does the line always have to be straight?” Then she drew a polygonal chain on the blackboard but not in the square. About half of the pupils were not able to apply the polygonal chain idea. Neither did she mention the middle-point. Pupils had to figure it out by themselves.

Danielle’s pupils worked in pairs. She walked around and commented on the pupils’ solutions: “Consider, whether there still could be some other alternatives” or: “Think.” She helped the pupils, for example by asking the whole class: “All your lines are straight. Should the line be straight?” In two pairs the pupils were able to apply this hint to their solutions.

Eve’s pupils sat in pairs, but everybody worked individually. Eve circulated giving positive comments but no guidance. When she noticed that most pupils had done the two basic cases, she referred to a jigsaw puzzle: “Did you think that your pieces are like pieces in a jigsaw puzzle?” She did not present any solution models but she encouraged some pupils to walk around to see other solutions. However, this did not help them further.

Gabrielle’s pupils also worked individually. Gabrielle guided her pupils by saying repeatedly: “You must think how you can get both parts similar, quite symmetrical.” About 25 minutes from the beginning, Gabrielle showed the curved-line models that she had obtained from Beatrice. Her inactivating guidance revealed the plot. Yet, most of her pupils did not benefit from this - perhaps because they were stuck with the idea of line symmetry. One pair had discovered the idea of curved line already earlier (level 3 solution). After Gabrielle's models, this pair produced a point-symmetrical solution (level 4) that was identical with the presented solution.

One teacher, Fatima, was classified to the No support category. Fatima's pupils worked individually on the task. While circulating, Fatima only looked at the pupils' work. She neither gave positive feedback nor guided the pupils, but let them work independently. When pupils asked her something she answered briefly like "Think for yourself" or "I won't give any more advice." Thus she gave neither cognitive nor emotional support for the pupils. However, also in her classroom the pupils were working hard and they were not afraid to risk making mistakes. At the end of the lesson, she showed the curved-line solutions that she had received from Beatrice. However, this did not help her pupils to go further.

Discussion and Summary Phase

This phase is important from the perspective of learning (cf. Polya, 1945). The extent of how well the learning will happen depends on the decisions the teacher makes in organising this phase. Three different ways for the summary phase could be distinguished in our data: Summary of solutions with discussions, Summary of solutions by presenting and No summary.

Summary of solutions with discussion. Danielle implemented the summary phase by gathering the pupils' solutions on the blackboard (see Figure 2). The pupils drew different solutions including curved-line and polygonal solutions. When the space on the blackboard was almost used up, some pupils began to draw their solutions in the same square. The pupils, and also their teacher, then suddenly observed that the middle-point of the square was in a key position: every solution line had to go through the middle-point. This emergent pattern is similar to the situation reported by Nunokawa (2006) and identified as an important cue for creative problem solving. Danielle's teaching action of gathering the solutions together seemed to help her pupils to get a good idea of the central features of the task. However, point-symmetry was not discussed.

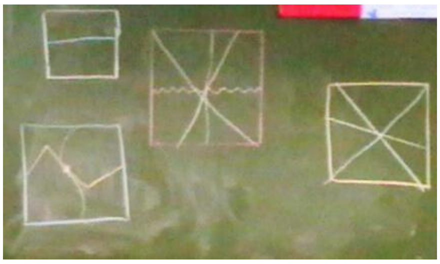

Figure 2. The pupils' solutions drawn on the blackboard during the summary phase.

Summary of solutions by presenting. Five teachers, Beatrice, Cecilia, Eve, Gabrielle and Fatima, let the pupils present their own solutions at the end of the lesson. In most cases, the pupils showed their solutions with the visualiser or hung them on the blackboard. These solutions were not commented upon any further. Cognitive support was quite modest but all these teachers gave much emotional support.

No summary. Ann had no separate summary phase but she discussed the solutions with her pupils during the lesson. However, only those solutions were discussed that Ann thought would show the pupils' thinking, such as Peter's solution (Figure 1). As homework, she asked the pupils to invent one different solution.

The summary of the teachers' actions in different phases is presented in Table 1. Each phase contains actions that might have had an influence on the pupils' performance. Also, the ideas that the teachers revealed to the pupils are included.

Table 1

Summary of the teachers' actions in the phases of launch, explore and discuss and summarise. The last column contains the ideas that the teachers revealed including the time when this occurred during the lesson.

	Launch	Explore	Discuss and summarise	Revealed ideas
Ann	Model	Activating support	No summary	
Beatrice	Model	Activating support	Solutions presented	Nature of the line (10 minutes from the beginning) Showed final models (10 minutes before the end)
Cecilia	Model	Commenting support	Solutions presented	Nature of the line (early in the explore phase)
Danielle	Model	Commenting support	Solutions with discussion	Nature of the line (10 minutes from the beginning)
Eve	No model	Commenting support	Solutions presented	Nature of the line (in the launch phase)
Fatima	Incorrect model	No support	Solutions presented	Showed final models (at the end of the lesson)
Gabrielle	Incorrect model	Commenting support	Solutions presented	Showed final models (in the middle of the lesson)

Pupils' Solutions

The pupils' performances in solving the problem were divided into five hierarchical levels (see Table 2). The level was determined based on the most sophisticated solution the pupil had invented during the problem-solving lesson.

The lowest level is No solution (Level 0). Only one pupil from the total of 86 was placed in this level. He had troubles in concentrating and he could not produce any solution.

In the Basic level (Level 1), the pupils found only the two obvious solutions (with a diagonal into two triangles and with a straight line parallel to the sides into two rectangles, see Figure 3, left example). In the next level, Straight line (Level 2), the pupil in addition to the two obvious solutions had divided the square with a straight line that started somewhere on the side of the square (see Figure 3, right example). To find such a solution requires some innovative thinking from the solver. There are infinitely many such solutions.

Figure 3. An example of Level 1 (line symmetry) and Level 2 (straight line) solutions.

In the third level, “Curved line,” the dividing line can be arbitrarily curved, a polygonal chain or curvilinear (see Figure 4). The number of these solutions is infinite.

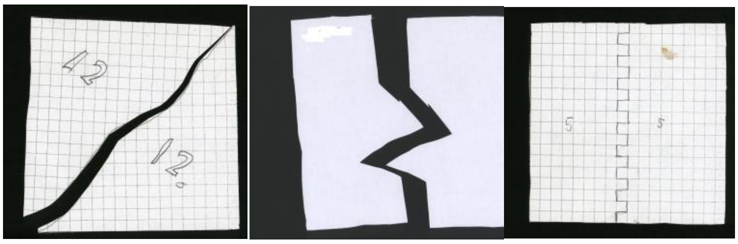

Figure 4. Three examples of Level 3 (curved line) solution (the numbers refer to the number of the solvers’ solutions).

The first two solutions in Figure 4 were classified to level 3, because the lines are of the right shape but they do not go through the middle-point. Also, the third solution was classified to level 3; although the middle-point has been marked in the figure, the line is not symmetric around it.

The highest level (Level 4) is represented by middle-point thinking, where the middle-point of the square is seen as the essential part of the solutions, so that the dividing lines – straight or curved – are symmetrical in relation to the middle-point (see Figure 5).

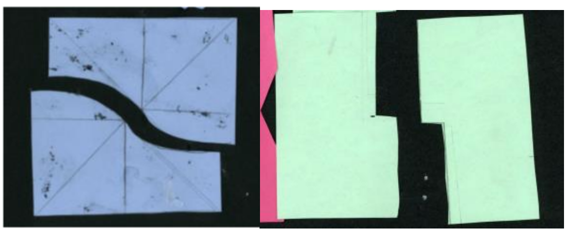

Figure 5. Two examples of Level 4 middle-point solution.

In the first picture in Figure 4 the classification is clear, as the pupil has drawn lines through the middle-point. In the other picture the pupil has marked the middle-point. When he cut out the pieces, they turned out a little bit different in size, but he obviously has understood the meaning of the middle-point. Therefore this solution was classified at level 4.

In Table 2, we present the distribution of pupils' solutions on different levels. Most of the pupils reached levels 1–3 in their solutions, but only 13 pupils (i.e. 15%) reached the highest level. The third graders were familiar with the idea of line symmetry (level 1). In order to reach levels 2–4, pupils needed to use new thinking in their solutions. Thus, we may say that about 61% of pupils were able to develop new ideas. The pupils returned a number of different solutions in levels 2, 3 and 4, but none of them referred to the infinite number of solutions in the group discussion.

Table 2

The distribution of the number of the pupils on different levels (N = 86).

No solution	Basic level	Straight line	Curved line	Middle-point
Level 0	Level 1	Level 2	Level 3	Level 4
1 (1%)	33 (38%)	21 (24%)	18 (21%)	13 (15%)

Pupils' Performance Related to the Teachers' Actions

We wanted to see if there was any variation in the pupils' solutions in different classes. Distribution of the pupils' solution levels in different teachers' classes is presented in Table 3.

Table 3

The distribution of the pupils' performances in percentages and scores in mathematics pre-test plus additional information according to each teacher (teacher's action during the

Launch phase: M = model, NoM = no model, IM = incorrect model; teacher's action during the Explore phase: AS = activating support, CS = commenting support, NS = no support; ; tools used during the lesson: S = scissors, BP = blank paper, GP = grid paper; teacher's way of revealing the solution: SM = solution model, L= line is not straight, NR = no revelation).

	Level of solution	Ann	Beatrice	Cecilia	Danielle	Eve	Fatima	Gabrielle
Pupils' performances	0	0%	0%	0%	8%	0%	0%	0%
	1	42%	14%	37%	0%	50%	86%	75%
	2	0%	29%	13%	58%	37%	0%	0%
	3	33%	14%	37%	34%	13%	14%	0%
	4	25%	43%	13%	0%	0%	0%	25%
Pre-test (max 15)								
class average		7.1	8.4	7.8	7.3	6.2	6.6	2.9
standard deviation		1.7	2.3	2.2	1.4	1.3	2.3	0.9
Launch phase		M	M	M	M	NoM	InM	InM
Explore phase		AS	AS	CS	CS	CS	NS	CS
Tools		S, GP	S, GP, BP	S, BP	S, BP	S, BP	S, BP	S, BP
Teacher revealed solutions		NR	L, SM	L	L	L	NR	SM

The teachers' connection to the pupils' level of solution seems to be important. Pupils in Ann's, Beatrice's and Cecilia's classes reached clearly higher-level performances (at least 50% at Levels 3–4) than the pupils in the other teachers' classes. Whereas at least half of the pupils' results in Eve's, Fatima's and Gabrielle's classes remained at Level 1 (variation 50%–86%).

Something in the teachers' actions must have caused this variation because results in the pre-test do not fully foretell the pupils' solution levels in this problem (see Table 3). There were no statistical differences between five best classes (Ann, Beatrice, Cecilia, Danielle and

Fatima) in the pre-test ($t(14) = 1.85$, $p = .086$ or weaker). However, pupils' solution levels in this problem vary much. Eve's class was statistically weaker ($t(16) = 2.36$, $p = .032$ or stronger) than the three best classes (Beatrice, Cesilia and Danielle) and Gabrielle's class was weakest compared to all the others ($t(5) = 3.02$, $p = .029$ or stronger). Therefore the results in Gabrielle's class could be expected to be weaker.

The way in which the teachers introduced the task seems to have a central consequence: Most of the pupils in Eve's (no model), Gabrielle's (incorrect model) and Fatima's (incorrect model) classes remained at level 1. It is also interesting to notice that all pupils except one reached at least level 2 in Danielle's class. It is possible that her circle model with many diagonals in the beginning of the lesson helped pupils to invent level 2 solutions. Also, the choice of tools influenced the results. Ann and Beatrice provided grid paper, and in these classes more students invented curved-lined middle-point solutions.

Also, the way in which the teachers guided their pupils is significant: Fatima did not guide at all and 86% of her pupils remained at level 1. Guidance with activating support was effective: The pupils in Ann's and Beatrice's classes invented more level 4 solutions than the pupils in other teachers' classes. Unfortunately Ann's class was restless and she was not able to guide all the pupils. Also, commenting support seems to help. Pupils perhaps try harder when the teacher encourages them and shows interest in their work.

Teachers' revelations about the solutions had very little effect. Beatrice and Gabrielle showed model solutions to the pupils, and during some part of the lesson, Beatrice, Cecilia, Danielle and Eve told the pupils that the line does not have to be straight. Based on the video recordings and the pupils' solutions we found that only in Gabrielle's class one pair of pupils used the level 4 solution that the teacher had shown. They had invented the level 3 solution by themselves. Apparently, pupils had difficulties to adopt a new idea from the teacher into their solutions. Perhaps the new solution was not within their zone of proximal development (c.f.

Vygotsky 1978). For the same reason the pupils gained little advantage from being allowed to walk around and to look at their classmates' solutions.

Discussion

A limitation of the study is that we could record pupils' problem solving process only partially and that we did not interview pupils or teachers after the lesson. Yet, our study was able to show much of the variation between different instructional decisions. The pupils' solution levels differed quite a lot between the classes and this seems to depend on the teachers' different actions during the problem-solving lesson. The ways in which the teacher introduced the task and guided her pupils seem to largely explain this difference in the solution levels. Especially the model used for task introduction seems to be important for inventing new ideas and Danielle's circular model helped pupils to open their thinking. Stein and her colleagues (2008) criticise the "first generation" of practice and research on problem solving, because then the teachers were expected to avoid any substantive guidance during the explore phase. Our results indicate that activating support produces good results. It seems to be important that the teacher is able to identify the critical points in pupils' solutions and base her guidance on those.

We were interested in how new mathematical ideas are developed in different classes. Hähkiöniemi and Leppäaho (2012) identified that prospective mathematics teachers often practice inactivating guidance and thus take away the opportunity of inventing the idea from the student. However, in our study the main problem for experienced elementary teachers seemed to be superficial guidance that did not help the pupil to find the idea. Also teachers in our study revealed solutions to the pupils, but not during personal guidance. Moreover, our results indicate that it was not enough to see or hear a solution to replicate the idea.

Why were some teachers guiding better than others? Why did some teachers present an improper model in the launch phase? One possible explanation is insufficient content knowledge (Shulman, 1986) about point symmetry. Based on their lesson plans and their guiding, Ann and Beatrice had clearly understood the meaning of the middle-point. It seems that the other teachers might not have been aware of the importance of the middle-point.

Another possibility is that some teachers did not take enough time to plan their lesson. How is this possible? Teacher manuals in Finland are easy to use and it does not require much effort to implement a lesson using the manual. Therefore, some teachers may not be used to preparing mathematics lessons on their own. However, in our project the aim was that the teachers would individually plan their lessons. Furthermore, most Finnish teachers are inexperienced in using open problems. Open problems require a different type of preparation and also different instructional approach (e.g. Sawada, 1997). The teacher should carefully familiarise herself with the task by solving it. It is important to find many different solutions in order to see the variety of the solutions. For example Gabrielle had prepared herself for the lesson. She had planned how to demonstrate the symmetry in the task. But maybe she had figured out only the two easiest solutions and considered the problem to be very simple.

Is it then necessary for the teacher to know all possible solutions? This is not a simple question. If the teacher knows the solutions she is able to pose good questions, understand pupils' difficulties and help them to go further. The negative option is that she may reveal the plot and does not let the pupils discover the solutions by themselves. Therefore it is also important to know about problem solving and understand the meaning of the different phases in the process. The importance of understanding the problem-solving process increases when the teacher does not know the solutions. A teacher who is accustomed to guiding problem solving knows certain types of good questions that will help the pupils move forward. She also knows that asking pupils to tell about their thinking often helps pupils to proceed. She knows the importance of emotional support. Guiding without preparation demands an open attitude and good content knowledge from the teacher. She must be able to listen empathically (cf. Pehkonen & Ahtee, 2006) and try to understand a pupil's ideas from the pupil's viewpoint. The teacher should be able to catch the pupil's idea and see its different connections in the task. With an open attitude and a good task, it is possible to reach good results as can be seen from the summary phase in Danielle's class.

It is also important to notice that the different possible aims in problem solving (Schroeder & Lester, 1989) lead to different conclusions regarding the phases of teaching. The summary phase is critical if the teacher aims to teach specific mathematical ideas via problem solving. However, if the main aim is to learn about problem solving, then the launch and explore phases are more important.

The pupils were later tested with open-ended tasks similar to the square problem. Our next step is to analyse how the pupils were able to use the knowledge acquired in this task. Additionally we want to see what kind of progress the teachers made in teaching problem solving.

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